Well-rounded equivariant deformation retracts of Teichmüller spaces

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Abstract

In this paper, we construct spines, i.e., Mod_g-equivariant deformation retracts, of the Teichmüller space $\mathcal{T}_g$ of compact Riemann surfaces of genus g. Specifically, we define a Mod_g-stable subspace S of positive codimension and construct an intrinsic Mod_g-equivariant deformation retraction from T_g to S. As an essential part of the proof, we construct a canonical Mod_g-deformation retraction of the Teichmüller space $\mathcal{T}_g$ to its thick part $\mathcal{T}_g(\varepsilon)$ when ε is sufficiently small. These equivariant deformation retracts of $\mathcal{T}_g$ give cocompact models of the universal space $\underline{E}\mathrm{Mod}_g$ for proper actions of the mapping class group Mod_g. These deformation retractions of $\mathcal{T}_g$ are motivated by the well-rounded deformation retraction of the space of lattices in $\mathbb{R}^n$.

1 Introduction

Let S_g be a compact oriented surface of genus g, and Mod_g be the mapping class group of S_g. Let $\mathcal{T}_g$ be the Teichmüller space of marked complex structures on S_g. We will assume that $g \geq 2$ in the following. Then $\mathcal{T}_g$ is also the moduli space of marked hyperbolic metrics on S_g.

It is known that $\mathcal{T}_g$ is a complex manifold diffeomorphic to $\mathbb{R}^{6g-6}$ and Mod_g acts holomorphically and properly on $\mathcal{T}_g$. It is also known that Mod_g contains torsion elements and does not act fixed point freely on $\mathcal{T}_g$. By using the geodesic convexity of the Weil-Petersson metric of $\mathcal{T}_g$ [Wo1] (or earthquakes in $\mathcal{T}_g$) and positive solutions of the Nielson realization problem [Ke] [Wo1], it can be shown [JW, Proposition 2.3] that $\mathcal{T}_g$ is a model of the universal space $\underline{E}\mathrm{Mod}_g$ of proper actions of Mod_g, which means that for every finite subgroup $F \subset \mathrm{Mod}_g$, the set of fixed points $(\mathcal{T}_g)^F$ is nonempty and contractible.

On the other hand, it is well-known that the quotient $\mathrm{Mod}_g \backslash \mathcal{T}_g$ is the moduli space of compact Riemann surfaces of genus g and is non-compact. For many applications, an important and natural problem is to find a model of the universal space $\underline{E}\Gamma$ for $\Gamma = \mathrm{Mod}_g$ which is Γ-cocompact, i.e., the quotient $\Gamma \backslash \underline{E}\Gamma$ is compact, or rather more to the point, is a finite CW-complex. Another closely related problem is to find a model of $\underline{E}\Gamma$ which is of as small dimension as possible, for example, equal to the virtual cohomological dimension of Γ.

For any $n \geq 1$, let $S_{g,n}$ be the surface obtained from S_g by removing n points, and $\mathcal{T}_{g,n}$ be the corresponding Teichmüller space of $S_{g,n}$ and $\mathrm{Mod}_{g,n}$ the corresponding mapping class group. Then $\mathcal{T}_{g,n}$ is also a model for the universal space $\underline{E}\mathrm{Mod}_{g,n}$ for proper actions of $\mathrm{Mod}_{g,n}$. It was shown

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in [BoE] [Pe2] that $\mathcal{T}_{g,n}$ admits the structure of $\mathrm{Mod}_{g,n}$-simplicial complex, and hence admits an equivariant deformation retraction to a subspace which is cofinite $\mathrm{Mod}_{g,n}$-CW complex of dimension equal to the virtual cohomological dimension of $\mathrm{Mod}_{g,n}$. This is a model of $\underline{E}\mathrm{Mod}_{g,n}$ of the smallest possible dimension. This result was used by Kontsevich [Ko] in proving a conjecture of Witten on intersection theory of the moduli space $\mathcal{M}_{g,n}$. The method for constructing the above subspace of $\mathcal{T}_{g,n}$ depends crucially on the assumption that $n \geq 1$ and cannot be applied to $\mathcal{T}_g$.

For the above problem to construct Mod_g-cocompact universal spaces $\underline{E}\mathrm{Mod}_g$, there are two approaches based on the action of Mod_g on $\mathcal{T}_g$: either construct a partial compactification $\overline{\mathcal{T}_g}$ such that the inclusion $\mathcal{T}_g \to \overline{\mathcal{T}_g}$ is a Mod_g-equivariant homotopy equivalence, or construct a Mod_g-stable subspace S such that $\mathrm{Mod}_g \backslash S$ is compact and there exists a Mod_g-equivariant deformation retraction from $\mathcal{T}_g$ to S. The second approach seems to be more accessible and might give spaces of smaller dimension than $\mathcal{T}_g$.

In a preprint [Th] circulated in 1985, Thurston proposed a candidate of Mod_g-equivariant deformation retract, i.e., a spine, of $\mathcal{T}_g$, of positive codimension. An outline was given to deform $\mathcal{T}_g$ to a small neighborhood of the proposed subspace. But the deformation retraction to the proposed subspace was not explicitly given, and there seems to be some other difficulties as well.

It is known [Ha] that the virtual cohomological dimension of Mod_g is $4g - 5$. An important problem is whether there exists a Mod_g-stable subspace of $\mathcal{T}_g$ which is of dimension $4g - 5$ and is a Mod_g-equivariant deformation retract of $\mathcal{T}_g$. The question whether such a deformation retract of $\mathcal{T}_g$ exists or not is Question 1.1 in [BV].

In this paper, we consider two spines of $\mathcal{T}_g$. The first one is the thick part $\mathcal{T}_g(\varepsilon)$ of $\mathcal{T}_g$, i.e., for any $\varepsilon > 0$ which is sufficiently small, $\mathcal{T}_g(\varepsilon)$ consists of hyperbolic surfaces which do not contain geodesics with length less than ε. The second subspace S consists of hyperbolic surfaces whose systoles, i.e., the shortest simple closed geodesics, contain at least an intersecting pair. (See Theorem 4.2 in §4 for more detail). An important point about the second spine S is that it is an *intrinsically* defined subspace of *positive* codimension.

It is known that $\mathcal{T}_g(\varepsilon)$ is a real analytic submanifold with corners and is stable under the action of Mod_g with compact quotient.

Existence of a Mod_g-equivariant deformation retraction of $\mathcal{T}_g$ to $\mathcal{T}_g(\varepsilon)$ was proved in [JW, Theorems 1.2 and 1.3]. Therefore, $\mathcal{T}_g(\varepsilon)$ is a Mod_g-cocompact $\underline{E}\Gamma$ space for $\Gamma = \mathrm{Mod}_g$.

On the other hand, the deformation retraction of $\mathcal{T}_g$ to $\mathcal{T}_g(\varepsilon)$ in [JW, §3] is the flow associated with a vector field which is patched up from local vector fields, which increase any fixed collection of short geodesics simultaneously, using a partition of unity. In order to get an equivariant deformation, the construction of the partition of unity is delicate. Since there is no intrinsic or canonical partition of unity, the deformation retraction is not unique or canonical.

A natural problem is to construct a deformation retraction of $\mathcal{T}_g$ to $\mathcal{T}_g(\varepsilon)$ which depends only on the intrinsic geometry of the hyperbolic surfaces in $\mathcal{T}_g$ and the geometry of $\mathcal{T}_g$. An answer is given in Theorem 3.9 below. Due to the intrinsic nature of the construction, it is automatically Mod_g-equivariant. Since the construction is motivated and similar to the well-rounded deformation retraction for the space of lattices in $\mathbb{R}^n$ [As1], which is explained in Remark 3.3 below, we also call it the *well-rounded deformation retraction* of the Teichmüller space $\mathcal{T}_g$ in the title.

The continuation of the deformation retraction to $\mathcal{T}_g(\varepsilon)$ gives rise to a deformation retraction to the second spine S. It is a real sub-analytic subspace of $\mathcal{T}_g$ of codimension at least 1. See Theorem 4.2 below for a precise statement.

It seems that this spine S is the *first example* of equivariant deformation retract of $\mathcal{T}_g$ which is of *positive codimension*. A natural problem is whether this idea can possibly be generalized to construct equivariant deformation retracts of $\mathcal{T}_g$ which are of higher codimension. This will depend on understanding subspaces of $\mathcal{T}_g$ consisting of hyperbolic surfaces whose systoles intersect. (See

[Sch] for a survey of some work of systoles of surfaces.) In Remark 4.3, we outline some arguments on how to obtain a spine of $\mathcal{T}_g$ of codimension at least 2.

Another natural problem is to find good candidates of spines of $\mathcal{T}_g$ which are of the optimal dimension $4g - 5$. It is reasonable to believe that spines of $\mathcal{T}_g$ should consist of "rounded hyperbolic surfaces", and hyperbolic surfaces in the spine S in Theorem 4.2 are in some sense the *least* rounded among all possible definitions of "rounded hyperbolic surfaces". One idea is to require hyperbolic surfaces to be cut into smaller pieces by some systoles such that the pieces are "rounded". Once good candidates are found, deformation retractions to them can be difficult.

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2   Definition of spines and examples

Let X be a topological space, and Γ a discrete group acting properly on X. A subset S of X is called an *equivariant spine* or simply a *spine* if

1. S is stable under Γ,

2. and there exists a Γ-equivariant deformation retraction from X to S.

For example, if Γ is a cofinite, nonuniform Fuchsian group, and $X = \mathbb{H}^2$ is the Poincare upper half-space, then $\mathbb{H}^2$ has an equivariant spine given by a tree. The best known example is $\Gamma = SL(2, \mathbb{Z})$. See [BoE]. (We note that when Γ is a torsion-free non-uniform Fuchsian group, then $\pi(\Gamma \backslash \mathbb{H}^2)$ is a free group.)

Proposition 2.1 *If X is contractible, then any spine S of X is also contractible. If X is a universal space for proper actions of Γ, then S is also a universal space for proper actions of Γ.*

Proof.   We only need to note that for any finite subgroup F of Γ, the fixed set S^F is a deformation retract of the fixed point set X^F in X and hence is nonempty and contractible.

In the following we assume that X is a universal space for proper actions of Γ. Then S is called a *minimal (or optimal) spine* if $\dim S = \text{vcd}\,\Gamma$, where vcd Γ is the virtual cohomological dimension of Γ. The reason is that since S is also a universal space for proper actions of Γ, $\dim S \geq \text{vcd}\,\Gamma$.

The spine S of X is called a *cofinite spine* if S is a Γ-CW complex and the quotient $\Gamma \backslash X$ is a finite CW-complex. S is called a *cocompact spine* if $\Gamma \backslash X$ is a compact space. We note that if S is cofinite, then it is cocompact. On the other hand, the converse is not automatically true, since a general Γ-space may not admit the structure of a Γ-CW-complex.

It is known that given any discrete group discrete Γ, there always exists a universal space $E\Gamma$ for proper and fixed point free actions of Γ, and a universal space $\underline{E}\Gamma$ for proper actions of Γ. Both $E\Gamma$ and $\underline{E}\Gamma$ are unique up to Γ-equivariant homotopy equivalence (see [Lu] and references there). The quotient $\Gamma \backslash E\Gamma$ is a classifying space $B\Gamma$ of Γ, i.e., $\pi_1(B\Gamma) = \Gamma$, and $\pi_i(B\Gamma) = \{1\}$ for $i \geq 2$. When Γ is torsion-free, then $\underline{E}\Gamma$ is equal to $E\Gamma$.

Models of $E\Gamma$ and $\underline{E}\Gamma$ can be constructed for groups by a general method due to Milnor and are infinite dimensional. But good models of $E\Gamma$ and $\underline{E}\Gamma$, in particular those having various finiteness properties, are important in order to understand finiteness properties of Γ such as finite generation, finite presentation and cohomological finiteness properties of Γ, and also for proofs of the Novikov

conjectures and the Baum-Connes conjecture for Γ. See [JW, §2] for an explanation of some applications.

For some basic groups such that arithmetic subgroups (or more general discrete subgroups) of Lie groups and mapping class groups, there are natural finite dimensional $\underline{E}\Gamma$-spaces. For the former groups, they are the symmetric spaces or more general contractible homogeneous spaces associated with the Lie groups, and for the latter groups, they are given by the Teichmüller spaces.

But such natural spaces are often not Γ-cofinite, or even Γ-cocompact, $\underline{E}\Gamma$-spaces, as pointed out in the introduction, and an important problem is to find good equivariant spines contained in them in order to construct cofinite models of $\underline{E}\Gamma$-spaces of dimension as small as possible.

Example. Suppose X is a simplicial complex with some faces of some simplexes missing. Let X^* be the completion of X, i.e, if an open simplex is contained in X^*, then all its simplicial faces are also contained in X^*. Suppose $X \neq X^*$. Then there is a canonical spine of X obtained as follows. Take the maximal full subcomplex of the barycentric subdivision of X^* that are disjoint from $X^* - X$, i.e., from the missing faces of X. This is the spine constructed in [As2] for the space of positive definite quadratic forms in n variables (or equivalently the space of lattices in $\mathbb{R}^n$), the Teichmüller space $\mathcal{T}_{g,n}$ of Riemann surfaces of genus g with n-punctures when $n > 0$ in [Ha], and the outer space associated with the out automorphism groups $Out(F_n)$ of the free groups in [CV].

On the other hand, if X does not have a structure of Γ-simplicial complex, it is often less clear how to construct a spine or whether a spine of positive codimension exists.

3 Deformation retraction of the Teichmüller space to the thick part and well-rounded lattices

Let S_g be a compact oriented surface of genus $g \geq 2$. A marked compact hyperbolic surface of genus g is a hyperbolic surface Σ_g together with a homotopy equivalence class $[\varphi]$ of diffeomorphisms $\varphi : \Sigma_g \to S_g$. Two marked hyperbolic surfaces $(\Sigma_{g,1}, [\varphi_1]), (\Sigma_{g,2}, [\varphi_2])$ are defined to be *equivalent* if there exists an isometry $h : \Sigma_{g,1} \to \Sigma_{g,2}$ such that $[\varphi_1] = [\varphi_2 \circ h] : \Sigma_{g,1} \to S_g$. Then the Teichmüller space $\mathcal{T}_g$ is the set of equivalence classes of marked compact hyperbolic surfaces of genus g:

$$\mathcal{T}_g = \{(\Sigma_g, [\varphi])\}/\sim .$$

Let $\mathrm{Diff}^+(S_g)$ be the group of orientation preserving diffeomorphisms of S_g, and $\mathrm{Diff}^0(S_g)$ the identity component of $\mathrm{Diff}^+(S_g)$. Then the quotient $\mathrm{Diff}^+(S_g)/\mathrm{Diff}^0(S_g)$ is the *mapping class group* Mod_d, and Mod_g acts on $\mathcal{T}_g$ by changing the markings of the marked hyperbolic surfaces.

By the collar theorem of hyperbolic surfaces, there exists a positive constant ε_0 such that for any compact hyperbolic surface Σ_g and any two closed geodesics γ_1, γ_2 in it, if

$$\ell(\gamma_1), \ell(\gamma_2) \leq \varepsilon_0,$$

then

$$\gamma_1 \cap \gamma_2 = \emptyset.$$

For any ε with $0 < \varepsilon \leq \varepsilon_0$, define the *$\varepsilon$-thick part* $\mathcal{T}_g(\varepsilon)$ by

$$\mathcal{T}_g(\varepsilon) = \{(\Sigma_g, [\varphi]) \mid \text{for all simple closed geodesic } \gamma \text{ in } \Sigma_g, \ \ell(\gamma) \geq \varepsilon\}.$$

Then the following result is known.

Proposition 3.1 *The subspace $\mathcal{T}_g(\varepsilon)$ is stable under Mod_g with a compact quotient $\mathrm{Mod}_g \backslash \mathcal{T}_g(\varepsilon)$. Under the above assumption that $0 < \varepsilon \leq \varepsilon_0$, $\mathcal{T}_g(\varepsilon)$ is a real analytic manifold with corners and hence admits a Mod_g-equivariant triangulation such that $\mathrm{Mod}_g \backslash \mathcal{T}_g(\varepsilon)$ is a finite CW-complex.*

Proof.   It is clear that $\mathcal{T}_g(\varepsilon)$ is stable under Mod_g since its definition does not depend on the markings. The compactness of the quotient $\mathrm{Mod}_g \backslash \mathcal{T}_g(\varepsilon)$ follows from the Mumford compactness criterion for subsets of $\mathrm{Mod}_g \backslash \mathcal{T}_g$ [Mu]. Near any boundary point $p = (\Sigma_g, [\varphi]) \in \partial \mathcal{T}_g(\varepsilon) - \mathcal{T}_g(\varepsilon)$, the subspace $\mathcal{T}_g(\varepsilon)$ is defined by the inequalities:

$$\ell(\gamma_1), \cdots, \ell(\gamma_k) \geq \varepsilon,$$

where $\gamma_1, \cdots, \gamma_k$ are all the simple closed geodesics on the marked surface Σ_g such that $\ell(\gamma_1)(p) = \varepsilon, \cdots, \ell(\gamma_k)(p) = \varepsilon$. By the assumption on ε, the geodesics $\gamma_1, \cdots, \gamma_k$ are disjoint. Then they can form a part of a collection of a pants decomposition of Σ_g, and their length functions are a part of the associated Fenchel-Nielsen coordinates, and hence their differentials $d\ell(\gamma_1), \cdots, d\ell(\gamma_k)$ are linearly independent. This implies that the subspace of $\mathcal{T}_g$ defined by he inequalities $\ell(\gamma_1) \geq \varepsilon, \cdots, \ell(\gamma_k) \geq \varepsilon$ is a real analytic submanifold of $\mathcal{T}_g$ with corners near the point $(\Sigma_g, [\varphi])$.

The main result of [JW, Theorems 1.2 and 1.3] is the following result.

Proposition 3.2 *For every ε with $0 < \varepsilon \leq \varepsilon_0$, there exists a Mod_g-equivariant deformation retraction of $\mathcal{T}_g$ to $\mathcal{T}_g(\varepsilon)$. In particular, $\mathcal{T}_g(\varepsilon)$ is a cofinite model of the universal space $\underline{E}\Gamma$ for proper actions of $\Gamma = \mathrm{Mod}_g$.*

The idea of the proof in [JW] is as follows. For any marked hyperbolic surface $(\Sigma_g, [\varphi])$ in the thin part $\mathcal{T}_g - \mathcal{T}_g(\varepsilon)$, we increase the lengths of the short geodesics by following the flow of a local vector field which is a suitable linear combination of the gradient vectors of the length functions of these short geodesics. Specifically, let $\gamma_1, \cdots, \gamma_k$ be all the short geodesics of Σ_g such that $\ell(\gamma_1) \leq \cdots \leq \ell(\gamma_k) \leq \varepsilon$. In a simple case, suppose that $\ell(\gamma_1) < \ell(\gamma_2)$. For any geodesic γ, let $\bigtriangledown \ell(\gamma)$ be the gradient of the function $\ell(\gamma)$ with respect to the Weil-Petersson metric of $\mathcal{T}_g$. Then the flow along the vector field $\bigtriangledown \ell(\gamma_1)$ will increase $\ell(\gamma_1)$ until it reaches $\ell(\gamma_2)$ or ε. The point to make use of the Weil-Petersson metric is that it is intrinsic and hence the flow is automatically Mod_g-equivariant.

On the other hand, a difficulty occurs if $\ell(\gamma_1) = \ell(\gamma_2)$ since it is not clear whether we should use either $\bigtriangledown \ell(\gamma_1)$ or $\bigtriangledown \ell(\gamma_2)$.

The way to solve this problem in [JW] consists of two steps: (1) introduce a local vector field near every point in the thin part $\mathcal{T}_g - \mathcal{T}_g(\varepsilon)$, which, in the notation above, is a suitable linear combination of $\bigtriangledown \ell(\gamma_1), \cdots, \bigtriangledown \ell(\gamma_k)$ on a small neighborhood of $(\Sigma_g, [\varphi])$ in $\mathcal{T}_g$ such that under its flow, the lengths of all the short geodesics $\gamma_1, \cdots, \gamma_k$ are increased simultaneously, (2) use a suitable Mod_g-invariant partition of unity to glue up the local vector fields to obtain a desired global vector field on the thin part $\mathcal{T}_g - \mathcal{T}_g(\varepsilon)$ that is *invariant* under Mod_g.

The construction of the partition of unity in step (2) is complicated, but not canonical or intrinsic. A natural problem is to obtain an equivariant deformation retraction of $\mathcal{T}_g$ to $\mathcal{T}_g(\varepsilon)$ which only depends on the intrinsic geometry of the hyperbolic surfaces Σ_g and the geometry of $\mathcal{T}_g$. The first purpose of this paper is to construct such an intrinsic equivariant deformation retraction. To do this, we first recall the well-rounded deformation retraction of lattices in $\mathbb{R}^n$.

Remark 3.3   Well-rounded deformation retraction of lattices. The pair $(\mathcal{T}_g, \mathrm{Mod}_g)$ has often be compared with the pair $(SL(n, \mathbb{R})/SO(n), SL(n, \mathbb{Z}))$ of a symmetric space $SL(n, \mathbb{R})/SO(n)$

of noncompact type and an arithmetic subgroup $SL(n, \mathbb{Z})$ acting on it. Unlike a general symmetric space, $SL(n, \mathbb{R})/SO(n)$ is the moduli space of marked unimodular lattices in $\mathbb{R}^n$, where a *marked lattice* is a lattice $\Lambda \subset \mathbb{R}^n$ together with an ordered basis $v_1, \cdots, v_n$ of Λ, and a lattice $\Lambda \subset \mathbb{R}^n$ is *unimodular* if $vol(\Lambda \backslash \mathbb{R}^n) = 1$. The locally symmetric space $SL(n, \mathbb{Z}) \backslash SL(n, \mathbb{R})/SO(n)$ is the moduli space of unimodular lattices in $\mathbb{R}^n$ up to isometry.

There is a known $SL(n, \mathbb{Z})$-equivariant deformation retraction of $SL(n, \mathbb{R})/SO(n)$ to the subspace of well-rounded lattices by successively scaling up the inner product on the linear subspace spanned by the shortest lattice vectors and hence increasing the length of shortest geodesics of the associated flat tori in order to reach more rounded flat tori. According to [As1], this result is due to Soule and Lannes and was presented in the unpublished thesis of Soule. A generalization of this method to the symmetric space associated with the general linear group of a division algebra over $\mathbb{Q}$ is given in [As1]. Since we only need the above special case $(SL(n, \mathbb{R})/SO(n), SL(n, \mathbb{Z}))$, we give a simplified summary of the deformation retraction in [As1] to motivate the deformation retraction of $\mathcal{T}_g$ in the next section.

More precisely, let $\mathbb{R}^n$ be given the usual Euclidean inner product $\langle, \rangle$, and $\Lambda \subset \mathbb{R}^n$ be a lattice. Let

$$m(\Lambda) = \inf\{\langle v, v \rangle \mid v \in \Lambda - 0\},$$

and

$$M(\Lambda) = \{v \in \Lambda \mid \langle v, v \rangle = m(\Lambda)\}.$$

If $M(\Lambda)$ spans $\mathbb{R}^n$, then the lattice Λ is called a *well-rounded lattice*. If $\Lambda \subset \mathbb{R}^n$ is a unimodular, not well-rounded lattice in $\mathbb{R}^n$, then it can be deformed *canonically* to a well-rounded unimodular lattice in several steps.

These notions can also be defined for marked lattices. Since there is a natural marking in the deformation, we suppress the marking in the following discussion. Or equivariantly, we are defining a deformation retraction of the locally symmetric space $SL(n, \mathbb{Z}) \backslash SL(n, \mathbb{R})/SO(n)$.

Suppose Λ is not a well-rounded lattice. Then the span $M(\Lambda) \otimes \mathbb{R}$, denoted by $V_M(\Lambda)$, is a proper linear subspace of $\mathbb{R}^n$. Let $V_M^\perp(\Lambda)$ be the orthogonal complement of $V_M(\Lambda)$ in $\mathbb{R}^n$. For any $t \geq 1$, define a new inner product $\langle, \rangle_t$ on $\mathbb{R}^n$ such that on $V_M(\Lambda)$, the inner product is scaled up by t, and on $V_M^\perp(\Lambda)$, it is scaled down by a unique factor so that with respect to the new inner $\langle, \rangle_t$ on $\mathbb{R}^n$, the lattice Λ is still a unimodular lattice. Note that this inner product depends on the lattice Λ. Scaling on the subspaces $V_M(\Lambda)$ and $V_M^\perp(\Lambda)$ in the opposite direction gives a canonical isometric identification between the two Euclidean spaces $(\mathbb{R}^n, \langle, \rangle_t)$ and $(\mathbb{R}^n, \langle, \rangle)$, and the image of Λ gives a new lattice Λ_t in the standard Euclidean space $(\mathbb{R}^n, \langle, \rangle)$.

If Λ is given a marking, i.e., an ordered basis $v_1, \cdots, v_n$, then the images of $v_1, \cdots, v_n$ in Λ_t form a basis of Λ_t as well. What is changed is the inner product and hence lengths of the vectors. This process also gives a canonical identification between marked lattices Λ_t and Λ.

In the deformation Λ_t, the monimal norm $m(\Lambda_t)$ is increasing. We deform Λ to Λ_t until $M(\Lambda_t)$ contains at least one more independent lattice vector, i.e., the dimension of $M(\Lambda_t) \otimes \mathbb{R}$ increases at least by 1 (note that for small values of t, $M(\Lambda_t)$ stays constant under the above identification between Λ_t and Λ). This finishes the first step of the deformation. Now we start again using the new vector subspace $V_M(\Lambda_t)$, and deform it again by increasing the norms of all *minimal lattice vectors simultaneously at the same rate*. After finitely many steps, $M(\Lambda_t)$ spans $\mathbb{R}^n$ and the lattice Λ_t is well-rounded.

Though the above deformation procedure is canonical, we still need to show that it gives a *continuous* map on $Y = SL(n, \mathbb{Z}) \backslash SL(n, \mathbb{R})/SO(n)$. To do this, we define a filtration of Y:

$$Y = Y_1 \supset Y_2 \cdots \supset Y_n,$$

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where $Y_j = \{\Lambda \in Y \mid \dim M(\Lambda) \otimes \mathbb{R} \geq j\}$, $j = 1, \cdots, n$. Clearly, $Y_1 = Y$, and Y_n is the subspace of well-rounded lattices. The complement $Y_{j-1} - Y_i$ consists of lattices whose associated subspace $M(\Lambda) \otimes \mathbb{R}$ has dimension equal to $j - 1$. The deformation retraction to the subspace Y_n of well-rounded lattices consists of composition of the deformation retractions of Y_{j-1} to Y_i for $j = 2, \cdots, n$. Hence it suffices to show that at every step, the retraction of Y_{j-1} to Y_j is continuous.

Let $\Lambda, \Lambda' \in Y_{j-1}$ be two lattices. If $\Lambda \in Y_{j-1} - Y_j$, we deform it to $\Lambda_t \in Y_j$ by the procedure described above; otherwise, $\Lambda \in Y_j$ and set $\Lambda_t = \Lambda$. Similar, we can define the deformation image $\Lambda'_{t'} \in Y_j$. We need to show that Λ_t and $\Lambda'_{t'}$ are close whenever Λ, Λ' are close.

There are two cases: (1) Suppose $\Lambda \in Y_j$. If $\Lambda' \in Y_j$, then $\Lambda'_{t'} = \Lambda'$ is close to $\Lambda_t = \Lambda$ by assumption. If $\Lambda' \in Y_{j-1} - Y_j$, then the next shortest norm of vectors in Λ' after $m(\Lambda')$ is close to $m(\Lambda')$. The reason is that $\dim M(\Lambda) \otimes \mathbb{R}$ is at least j but $\dim M(\Lambda') \otimes \mathbb{R} = j - 1$. This implies that the stretching factor in reaching $\Lambda'_{t'}$ from Λ' is close to 1, and $\Lambda'_{t'}$ is close to Λ' and hence close to $\Lambda_t = \Lambda$.

(2) Suppose that $\Lambda \in Y_{j-1} - Y_j$. Then $\dim M(\Lambda) \otimes \mathbb{R} = j - 1$. We claim that when Λ' is close enough to Λ, then $\dim M(\Lambda') \otimes \mathbb{R} = j - 1$. By assumption, for any $v \in \Lambda - M(\lambda) - \{0\}$, $\|v\| > m(\Lambda)$, and hence these exists a positive number ε such that for all $v \in \Lambda - M(\lambda) - \{0\}$, $\|v\| \geq m(\Lambda) + \varepsilon$. Then for any $\Lambda' \in Y_{j-1}$ close to Λ, there exists $g \in SL(n, \mathbb{R})$ close to the identity element such that $\Lambda' = g\Lambda$. This implies that only minimal vectors in $M(\Lambda)$ can be mapped to $M(\Lambda')$, i.e., $M(g\Lambda) \subseteq gM(\Lambda)$, and hence $\dim M(\Lambda') \otimes \mathbb{R} \leq \dim M(\Lambda) \otimes \mathbb{R}$. Since $\Lambda' \in Y_{j-1}$, $\dim M(\Lambda') \otimes \mathbb{R} \geq j - 1$. By assumption, $\dim M(\Lambda) \otimes \mathbb{R} = j - 1$, it follows that $\dim M(\Lambda') \otimes \mathbb{R} = \dim M(\Lambda) \otimes \mathbb{R}$.

Since $m(\Lambda)$ and $m(\Lambda')$ are close and the next smallest norms in Λ, Λ' are also close, the equality of the dimension $\dim M(\Lambda') \otimes \mathbb{R} = \dim M(\Lambda) \otimes \mathbb{R}$ implies that the scaling factor t needed for $M(\Lambda_t)$ to have a higher dimension, i.e., for Λ_t to reach Y_j is close to the scaling factor t' needed for $\Lambda'_{t'}$ to reach Y_j. This implies that Λ_t and $\Lambda'_{t'}$ are close. This completes the proof of the continuity of the deformation retraction from Y_{j-1} to Y_j, and hence of the deformation retraction of Y to the subspace Y_n of well-rounded lattices.

A tempting idea is to carry out the same deformation for Teichmüller spaces by increasing the lengths of shortest geodesics while decreasing the lengths of other geodesics. But there are no linear structures and orthogonal complement on the set of closed geodesics as in the case of lattices in $\mathbb{R}^n$, and it is not clear whether such a deformation is possible. We will need to deform differently.

For any positive $\varepsilon \leq \varepsilon_0$, decompose the thin part $\mathcal{T}_g - \mathcal{T}_g(\varepsilon)$ into a disjoint union of submanifolds according to the multiplicity of the shortest geodesics, or *systoles*.

Every simple closed curve c of the base surface S_g which is not homotopic to a point induces a unique simple closed geodesic in every marked hyperbolic surface $(\Sigma_g, [\varphi])$, which is contained in the homotopy class $[\varphi^{-1}(c)]$ of simple closed curves.

For a collection of pairwise disjoint simple closed curves $c_1, \cdots, c_k$ of S_g, let $\gamma_1, \cdots, \gamma_k$ be the corresponding geodesics in $(\Sigma_g, [\varphi])$. Define a subspace

$$\mathcal{T}_{g,c_1,\cdots,c_k} = \{(\Sigma_g, [\varphi]) \mid \ell(\gamma_1) = \cdots = \ell(\gamma_k) < \ell(\gamma), \text{ for any other simple closed geodesic } \gamma \subset \Sigma\}.$$

In terms of systoles, $\mathcal{T}_{g,c_1,\cdots,c_k}$ consists of hyperbolic surfaces whose systoles are disjoint simple closed geodesics $\gamma_1, \cdots, \gamma_k$.

Proposition 3.4 *For every collection of disjoint simple closed curves $c_1, \cdots, c_k$, the index k satisfies the bound: $k \leq 3g - 3$. The intersection $(\mathcal{T}_g - \mathcal{T}_g(\varepsilon)) \cap \mathcal{T}_{g,c_1,\cdots,c_k}$ is a nonempty real analytic submanifold. The thin part $\mathcal{T}_g - \mathcal{T}_g(\varepsilon)$ admits a Mod_g-equivariant **disjoint decomposition** into*

$(\mathcal{T}_g - \mathcal{T}_g(\varepsilon)) \cap \mathcal{T}_{g,c_1,\cdots,c_k}$, when $\{c_1, \cdots, c_k\}$ ranges over all possible collections of disjoint simple closed curves of the base surface S_g.

Proof. The first statement is standard that the maximum number of disjoint, simple closed geodesics in every hyperbolic surface Σ_g is equal to $3g - 3$. The second statement follows from the proof of Proposition 3.1. and the fact that for any collection of disjoint simple closed curves $c_1, \cdots, c_k$, there are marked hyperbolic surfaces whose corresponding simple closed geodesics have arbitrarily short lengths.

The third statement follows from the fact that for any hyperbolic surface Σ_g, the lengths of its simple closed geodesics form an increasing sequence with finite multiplicity going to infinity, and hence the surface Σ_g belongs to some subspace $\mathcal{T}_{g,c_1,\cdots,c_k}$, where $c_1, \cdots, c_k$ are disjoint since their lengths are less than ε.

For each collection $c_1, \cdots, c_k$ of disjoint, simple closed curves of S_g, we define a vector field $V_{c_1,\cdots,c_k}$ on the associated subspace $(\mathcal{T}_g - \mathcal{T}_g(\varepsilon)) \cap \mathcal{T}_{g,c_1,\cdots,c_k}$ as follows. Since $(\mathcal{T}_g - \mathcal{T}_g(\varepsilon)) \cap \mathcal{T}_{g,c_1,\cdots,c_k}$ is a submanifold of $\mathcal{T}_g$, the Weil-Petersson metric of $\mathcal{T}_g$ restricts to a Riemannian metric on it. By definition, the length functions $\ell(\gamma_1), \cdots, \ell(\gamma_k)$ are equal on $\mathcal{T}_{g,c_1,\cdots,c_k}$ and hence define a common function, denoted by ℓ. Let $\triangledown \ell^{1/2}$ be the gradient of $\ell^{1/2}$ with respect to the restricted Riemannian metric on $(\mathcal{T}_g - \mathcal{T}_g(\varepsilon)) \cap \mathcal{T}_{g,c_1,\cdots,c_k}$.

Lemma 3.5 *In the above notation, the function ℓ has no critical point on $(\mathcal{T}_g - \mathcal{T}_g(\varepsilon)) \cap \mathcal{T}_{g,c_1,\cdots,c_k}$ and the vector field $\triangledown \ell^{1/2}$, denoted by $V_{c_1,\cdots,c_k}$, does not vanish at any point on $(\mathcal{T}_g - \mathcal{T}_g(\varepsilon)) \cap \mathcal{T}_{g,c_1,\cdots,c_k}$. Furthermore, $\triangledown \ell^{1/2}$ is uniformly bounded away from 0 and from above.*

Proof. Since the geodesics $\gamma_1, \cdots, \gamma_k$ are disjoint, their length functions $\ell_1, \cdots, \ell_k$ appear as a part of the Fenchel-Nielsen coordinates associated with a collection of maximal disjoint simple closed geodesics. This implies that $\triangledown \ell_i \neq 0$ for $i = 1, \cdots, k$. By [Wo3, Lemma 3.12], for any two disjoint geodesics γ_i, γ_j, $\langle \triangledown \ell_i, \triangledown \ell_j \rangle > 0$. This implies that on $\mathcal{T}_{g,c_1,\cdots,c_k}$, which can be thought of a partial diagonal, $\triangledown \ell \neq 0$ at every point. This implies that $\triangledown \ell^{1/2} \neq 0$ too. When ℓ is small, the uniform boundedness of $\triangledown \ell^{1/2}$ follows from [Wo3, Lemma 3.12] (also [JW, equation (3.1)]).

Remark 3.6 We note that an important reason for using $\triangledown \ell^{1/2}$ instead of $\triangledown \ell$ is that the former is uniformly bounded away from 0 and from above when ℓ belongs to $(0, a]$ for any $a > 0$, in particular near 0. See the discussion on [Wo2, page 278].

Lemma 3.7 *The vector fields $V_{c_1,\cdots,c_k}$ together define a Mod_g-equivariant vector field on the thin part $\mathcal{T}_g - \mathcal{T}_g(\varepsilon)$. Denote this vector field by V,*

Proof. By Proposition 3.4, the thin part $\mathcal{T}_g - \mathcal{T}_g(\varepsilon)$ admits a Mod_g-equivariant *disjoint* decomposition into $(\mathcal{T}_g - \mathcal{T}_g(\varepsilon)) \cap \mathcal{T}_{g,c_1,\cdots,c_k}$. Therefore, the vector fields $V_{c_1,\cdots,c_k}$ combine and define a vector field on $\mathcal{T}_g - \mathcal{T}_g(\varepsilon)$. Since the submanifolds and the vector fields are defined intrinsically in terms of the length functions and the Weil-Petersson metric, it is clear that V is equivariant with respect to Mod_g.

Remark 3.8 We note that the vector field V is *not continuous* in general. For example, $\mathcal{T}_{c_1,c_2}$ is contained in the closure of both $\mathcal{T}_{c_1}$ and $\mathcal{T}_{c_2}$. The vector fields V_{c_1} and V_{c_2} will both extend continuously to $\mathcal{T}_{c_1,c_2}$ but have different values at these boundary points. The vector field V_{c_1,c_2} is an average of V_{c_1} and V_{c_2}. The same pheonmenon of discontinuity occurs in the well-rounded

deformation of lattices in $SL(n, \mathbb{R})/SO(n)$ in Remark 3.3. But the deformation paths are continuous. In some sense, it amounts to the fact that the integral of a piecewise continuous function is continuous.

Theorem 3.9 *For every ε with $0 < \varepsilon \leq \varepsilon_0$, there exists an intrinsic Mod_g-equivariant deformation retraction of $\mathcal{T}_g$ to $\mathcal{T}_g(\varepsilon)$. In particular, $\mathcal{T}_g(\varepsilon)$ is a cofinite model of the universal space $\underline{E}\Gamma$ for proper actions of $\Gamma = \mathrm{Mod}_g$.*

Proof. The deformation retraction is the flow of the vector field V defined in Lemma 3.7. Though the vector field V is not continuous, it does not cause any problem and the flow is still continuous. Roughly, for any hyperbolic surface in the thin part $\mathcal{T}_g - \mathcal{T}_g(\varepsilon)$, we increase the lengths of the systoles at the same rate until we have reached the thick part $\mathcal{T}_g(\varepsilon)$ or the systolic length equals to the length of the next shortest geodesic, i.e, the systoles have included more geodesics, and then we repeat the above procedure. It will reach the thick part and stop after finitely many steps.

Specifically, we deform any marked hyperbolic surface $(\Sigma_g, [\varphi])$ in the thin part $\mathcal{T}_g - \mathcal{T}_g(\varepsilon)$ as follows. Let $\ell(\gamma_1) \leq \ell(\gamma_2) \leq \cdots \ell(\gamma_n) \leq \cdots$ be the lengths of its simple closed geodesics arranged in the increasing order.

Suppose $\ell(\gamma_1) = \ell(\gamma_2) = \cdots = \ell(\gamma_k) < \ell(\gamma_{k+1})$, i.e., the systoles consist of $\gamma_1, \cdots, \gamma_k$. This means that $(\Sigma_g, [\varphi]) \in \mathcal{T}_{g, c_1, \cdots, c_k}$. Then we increase the lengths $\ell(\gamma_1), \cdots, \ell(\gamma_k)$ simultaneously at the same rate, i.e., we keep the deformation path inside $\mathcal{T}_{g, c_1, \cdots, c_k}$. This can be achieved by the flow of the nowhere vanishing vector field $V_{c_1, \cdots, c_k}$ on the submanifold $(\mathcal{T}_g - \mathcal{T}_g(\varepsilon)) \cap \mathcal{T}_{g, c_1, \cdots, c_k}$ until the length of the systoles reach the next length $\ell(\gamma_{k+1})$ or the value ε, i.e., the surface has reached a point of the thick part $\mathcal{T}_g(\varepsilon)$.

Suppose that the deformation has not reached the thick part $\mathcal{T}_g(\varepsilon)$ yet. At the next step, we have $\ell(\gamma_1) = \ell(\gamma_2) = \cdots = \ell(\gamma_{k'}) < \ell(\gamma_{k'+1})$, where $k' \geq k + 1$. Since $\ell(\gamma_1) = \ell(\gamma_2) = \cdots = \ell(\gamma_{k'}) < \varepsilon$, $\gamma_1, \cdots, \gamma_{k'}$ are disjoint. We can deform as above using the vector field $V_{c_1, \cdots, c_{k'}}$ in order to increase the length of the systoles at the same rate and stop if the systolic length reaches the length of the next shortest geodesic or the surface has reached a point of the thick part. Since we have reached $\mathcal{T}_g(\varepsilon)$ already whenever $k' > 3g - 3$, this process will terminate after at most $3g - 3$ steps.

To show that the deformation retraction is continuous, we follow the proof of the continuity of the well-rounded deformation retraction of lattices in $\mathbb{R}^n$ recalled in Remark 3.3.

For $j = 1, \cdots, 3g - 3$, let $\mathcal{T}_g^j$ be the subspace of $\mathcal{T}_g$ of marked hyperbolic surfaces whose systoles contain at least j disjoint simple geodesics. Then $\mathcal{T}_g^1 = \mathcal{T}_g$. Define $\mathcal{T}_g^{3g-2} = \emptyset$. For each $j = 2, \cdots, 3g - 2$, the above discussion gives a deformation retraction of $\mathcal{T}_g^{j-1} \cup \mathcal{T}_g(\varepsilon)$ to $\mathcal{T}_g^j \cup \mathcal{T}_g(\varepsilon)$, and their composition gives the deformation retraction to the thick part $\mathcal{T}_g(\varepsilon)$. Therefore, it suffices to show that the deformation retraction at each step, from $\mathcal{T}_g^{j-1} \cup \mathcal{T}_g(\varepsilon)$ to $\mathcal{T}_g^j \cup \mathcal{T}_g(\varepsilon)$, is continuous.

Fix a $j \in 1, \cdots, 3g - 2$. Let $(\Sigma_g, [\varphi]), (\Sigma'_g, [\varphi]')$ be two points in $\mathcal{T}_g^{j-1} \cup \mathcal{T}_g(\varepsilon)$. Denote their deformed image in $\mathcal{T}_g^j \cup \mathcal{T}_g(\varepsilon)$ by $(\Sigma_g, [\varphi])_t$ and $(\Sigma'_g, [\varphi]')_{t'}$. (The subscripts indicate the times needed for the flow). We need to show that when $(\Sigma_g, [\varphi]), (\Sigma'_g, [\varphi]')$ are close, then $(\Sigma_g, [\varphi])_t$, $(\Sigma'_g, [\varphi]')_{t'}$ are also close. There are two cases to consider.

Case (1): $(\Sigma_g, [\varphi]) \in \mathcal{T}_g^j \cup \mathcal{T}_g(\varepsilon)$. Then $(\Sigma_g, [\varphi])_t = (\Sigma_g, [\varphi])$. If $(\Sigma'_g, [\varphi']) \in \mathcal{T}_g^j \cup \mathcal{T}_g(\varepsilon)$, then $(\Sigma'_g, [\varphi'])_{t'} = (\Sigma'_g, [\varphi'])$ is close to $(\Sigma_g, [\varphi])_t$.

Otherwise, $(\Sigma'_g, [\varphi']) \in (\mathcal{T}_g^{j-1} - \mathcal{T}_g^j) \cap \mathcal{T}_g - \mathcal{T}_g(\varepsilon)$, and the systoles of $(\Sigma'_g, [\varphi'])$ consists of $j - 1$ geodesics. Since $(\Sigma'_g, [\varphi'])$ is close to $(\Sigma_g, [\varphi])$, and the systoles of $(\Sigma_g, [\varphi])$ consists of at least j geodesics, it imples that the length of jth shortest geodesic of $(\Sigma'_g, [\varphi'])$ is close to its systolic length, and it takes a small deformation for $(\Sigma'_g, [\varphi'])_{t'}$ to reach a point in $\mathcal{T}_g^j \cup \mathcal{T}_g(\varepsilon)$. (Here we have

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used the fact that by Lemma 3.5, each vector field $V_{c_1, \cdots, c_k}$ is continuous and its norm is uniformly bounded from both below and above.) Therefore, $(\Sigma_g, [\varphi])_t$, $(\Sigma'_g, [\varphi]')_{t'}$ are also close.

Case (2): $(\Sigma_g, [\varphi]) \in (\mathcal{T}_g^{j-1} - \mathcal{T}_g^j) \cap \mathcal{T}_g - \mathcal{T}_g(\varepsilon)$. Then the systoles of $(\Sigma_g, [\varphi])$ consist of $j-1$ geodesics, $\gamma_1, \cdots, \gamma_{j-1}$, which correspond to simple closed curves $c_1, \cdots, c_{j-1}$ of the base surface S_g. We claim that when $(\Sigma'_g, [\varphi'])$ is sufficiently close to $(\Sigma_g, [\varphi])$, then the systoles of $(\Sigma'_g, [\varphi'])$ also consist of $j-1$ geodesics $\gamma'_1, \cdots, \gamma'_{j-1}$ which correspond to the same set of simple closed curves $c_1, \cdots, c_{j-1}$ of the base surface S_g. To prove the claim, we note that there exists a positive number δ such that for every simple closed geodesic γ of $(\Sigma_g, [\varphi])$ different from $\gamma_1, \cdots, \gamma_{j-1}$, $\ell(\gamma) \geq \ell(\gamma_1) + \delta$. This implies that when $(\Sigma'_g, [\varphi'])$ is sufficiently close to $(\Sigma_g, [\varphi])$, only geodesics of $(\Sigma'_g, [\varphi'])$ correspond to the simple closed curves $c_1, \cdots, c_{j-1}$ on the base surface S_g can be systoles. Since $(\Sigma'_g, [\varphi']) \in \mathcal{T}_g^{j-1}$, it must have at least $j-1$ systoles. This implies that it has exactly $j-1$ systoles and the claim is proved.

By the claim, $(\Sigma_g, [\varphi])$, $(\Sigma'_g, [\varphi'])$ belong to the same submanifold $\mathcal{T}_{g, c_1, \cdots, c_k}$. Then under the flow defined by the vector field $V_{c_1, \cdots, c_k}$, they reach their deformation points $(\Sigma_g, [\varphi])_t$, $(\Sigma'_g, [\varphi'])_{t'}$ in $\mathcal{T}_g^j \cup T_g(\varepsilon)$. We note that by Lemma 3.5, each vector field $V_{c_1, \cdots, c_k}$ is continuous and its norm is uniformly bounded from both below and above, and hence the time it takes to move any point of $(\mathcal{T}_g^{j-1} - \mathcal{T}_g(\varepsilon)) \cap \mathcal{T}_{g, c_1, \cdots, c_k}$ to $(\mathcal{T}_g^j \cup \mathcal{T}_g(\varepsilon))$ is also uniformly bounded in terms of its distance to $(\mathcal{T}_g^j \cup \mathcal{T}_g(\varepsilon))$ and depends continuously on the initial point. Therefore, when $(\Sigma_g, [\varphi])$, $(\Sigma'_g, [\varphi'])$ are close, $(\Sigma_g, [\varphi])_t$, $(\Sigma'_g, [\varphi'])_{t'}$ are also close, and the continuity of the deformation retraction of $\mathcal{T}_g^{j-1}$ to $\mathcal{T}_g^j$ is proved.

Since the flow at every step and hence the whole flow from $\mathcal{T}_g$ to $\mathcal{T}_g(\varepsilon)$ is intrinsically defined and hence equivariant with respect to Mod_g, this completes the proof of Theorem 3.9.

4 Deformation of Teichmüller space to a spine of positive codimension

Though the thick part $\mathcal{T}_g(\varepsilon)$ of $\mathcal{T}_g$ gives a cofinite model of the universal space $\underline{E}\Gamma$ for $\Gamma = \mathrm{Mod}_g$, it is a subspace of $\mathcal{T}_g$ of codimension 0.

It is tempting to conjecture that $\mathcal{T}_g$ admits a Mod_g-equivariant deformation retraction to a subspace of dimension equal to $4g - 5$, which is equal to the virtual cohomological dimension of Mod_g.

One modest step towards this is to construct subspaces of $\mathcal{T}_g$ which are of positive codimension and equivariant deformation retracts of $\mathcal{T}_g$ to them. As mentioned in the introduction, such an attempt was first made in [Th]. In this section, we continue the flow in the previous section and deform $\mathcal{T}_g(\varepsilon)$ (or rather $\mathcal{T}_g$) to a subspace of positive codimension.

For any marked hyperbolic surface $(\Sigma_g, [\varphi])$, arrange its lengths of simple closed geodesics in the increasing order:

$$\ell(\gamma_1) \leq \ell(\gamma_2) \leq \cdots \leq \ell(\gamma_n) \leq \cdots .$$

Define a *well-rounded subspace* $S \subset \mathcal{T}_g$ to consist of marked hyperbolic surfaces $(\Sigma_g, [\varphi])$ satisfying the conditions:

1. $\ell(\gamma_1) = \cdots = \ell(\gamma_k) < \ell(\gamma_{k+1})$ for some $k \geq 2$.

2. some pairs of geodesics from $\gamma_1, \cdots, \gamma_k$, i.e., some paris of systoles of $(\Sigma_g, [\varphi])$, intersect each other.

Proposition 4.1 *The well-rounded subspace S is stable under Mod_g with a compact quotient, and the codimension of S in $\mathcal{T}_g$ is positive. Furthermore, S is a subanalytic subspace and hence admits a Mod_g-equivariant triangulation such that the quotient $\mathrm{Mod}_g\backslash S$ is a finite CW-complex.*

Proof. It is clear that S is stable under Mod_g since it is defined in terms of the lengths of closed geodesics of hyperbolic surfaces in $\mathcal{T}_g$. For any hyperbolic surface in S, since two of the shortest geodesics intersect, by the collar theorem, their length is uniformly bounded from below by a constant which depends only on g. Then by the Mumford compactness criterion [Mu], the quotient $\mathrm{Mod}_g\backslash S$ is compact.

Near any point in S, S is locally defined by at least one real analytic equation, $\ell(\gamma_1) = \ell(\gamma_2), \cdots, \ell(\gamma_{k-1}) = \ell(\gamma_k)$. This implies that S is of positive codimension.

Since the geodesic length functions are real analytic, S is a sub-analytic space. The existence of equivariant triangulation of S follows from a general result on existence of equivariant triangulation.

The second result of this paper is to show that S is an equivariant deformation retraction of $\mathcal{T}_g$.

Theorem 4.2 *The subspace S is a cofinite, equivariant spine of $\mathcal{T}_g$ of positive codimension with respect to Mod_g.*

Proof. For any collection of simple closed curves $c_1, \cdots, c_k$ of the base surface S_g, by the same proof of Proposition 3.4, we can show that the subspace $\mathcal{T}_{g,c_1,\cdots,c_k} \cap (\mathcal{T}_g - S)$ associated with it is a smooth submanifold, and $\mathcal{T}_g - S$ admits a Mod_g-equivariant decomposition into disjoint submanifolds $\mathcal{T}_{g,c_1,\cdots,c_k} \cap (\mathcal{T}_g - S)$ as in the case of $\mathcal{T}_g - \mathcal{T}_g(\varepsilon)$.

We also note that as in Lemma 3.5 the disjointness of the simple closed curves $c_1, \cdots, c_k$ implies that $k \leq 3g - 3$, and the vector field $V_{c_1,\cdots,c_k} = \bigtriangledown\ell^{1/2}$ is defined on $\mathcal{T}_{g,c_1,\cdots,c_k} \cap (\mathcal{T}_g - S)$ and is continuous and its norm is bounded away from 0 and from above. To prove this, we note that the norm of $\bigtriangledown\ell^{1/2}$ is uniformly bounded away from zero and the above when $\ell \in (0, a]$, where a is any positive constant. The condition $\ell \in (0, a]$ for some $a > 0$ is satisfied since ℓ is the systole of the hyperbolic surface.

Then the same proof of Theorem 3.9 works by replacing $\mathcal{T}_g(\varepsilon)$ by S, and Theorem 4.2 can be proved.

Remark 4.3 One natural question is whether the spine S in Theorem 4.2 can be further deformation retracted to a subspace of smaller dimension. If S were a smooth manifold, then the above flow might be continued. In general S should be a singular subspace.

Define two subspaces of S by

$$S' = \{(\Sigma_g, [\varphi]) \in S \mid \text{ there are exactly two systoles } \gamma_1, \gamma_2\}, \quad \text{and} \quad S'' = S - S'.$$

It is clear that each hyperbolic surface in S'' contains at least three systoles, and hence S'' is of codimension at least 2.

Next we outline arguments from Wolpert and Parlier which show that S' is smooth, and $\bigtriangledown\ell$ is a nowhere vanishing vector field on S' such that its flow defines a deformation retraction of S to S''.

Briefly, we note that for each hyperbolic surface in S', the two systoles γ_1, γ_2 intersect at one point. Using the fact that the difference of gradients of the length functions of two geodesics intersecting at a single point is never zero, in particular $\bigtriangledown\ell(\gamma_1) - \bigtriangledown\ell(\gamma_2) \neq 0$, we conclude that S' is a smooth submanifold of $\mathcal{T}_g$.

Let $\ell = \ell(\gamma_1) = \ell(\gamma_2)$ be the systole function on S'. Let $\bigtriangledown\ell$ be the gradient of ℓ on S' with respect to the restriction of the Weil-Petersson metric of $\mathcal{T}_g$. If $\bigtriangledown\ell$ is not zero, then $\bigtriangledown\ell$ is the direction along which both $\ell(\gamma_1), \ell(\gamma_2)$ are increased at the maximal rate while the equality $\ell(\gamma_1) = \ell(\gamma_2)$ is preserved.

Since γ_1, γ_2 are two intersecting systoles, it can be shown that γ_1, γ_2 do not fill Σ_g. For example, when $g = 2$, then the complement $\Sigma_g - \gamma_1 - \gamma_2$ is a one holed torus. For $g \geq 2$, the complement $\Sigma_g - \gamma_1 - \gamma_2$ is a genus $(g-1)$-surface with one boundary component.

Let δ be a simple closed geodesic which is disjoint from γ_1, γ_2. By [Wo3, Lemma 3.12], a deformation in $\mathcal{T}_g$ along the direction of $\bigtriangledown\ell(\delta)$ will increase both $\ell(\gamma_1), \ell(\gamma_2)$. This implies that $\bigtriangledown\ell$ is nonzero. Then the proof of Theorem 4.2 (or Theorem 3.9) can be repeated to show that the flow of $\bigtriangledown\ell$ defines a deformation retraction of S to S''. Therefore $\mathcal{T}_g$ admits an equivariant deformation retract S'' of codimension at least 2.

It seems very difficult that this deformation retract S'' can be pushed further to construct a spine of $\mathcal{T}_g$ with higher codimension. For example, it is not clear whether the subspace S''' of S'' consisting of hyperbolic surfaces with *exact three systoles* is a smooth submanifold of $\mathcal{T}_g$. When $g = 3$, S''' is of the optimal dimension, i,e., the virtual cohomological dimension of Mod_g.

On the other hand, it might be helpful to note that [Wo3, Lemma 3.12] implies the following result: *If a collection of systoles* $\gamma_1, \cdots, \gamma_k$ *of a marked hyperbolic surface* Σ_g *is not filling, then a deformation in* $\mathcal{T}_g$ *along the direction of* $\bigtriangledown\ell(\delta)$, *the gradient of the length of a disjoint simple closed geodesic* δ *of* Σ_g, *increases simultaneously lengths of all the systoles* $\gamma_1, \cdots, \gamma_k$. *Of course, the equality* $\ell(\gamma_1) = \cdots = \ell(\gamma_k)$ *may not be preserved any more, i.e., the deformed geodesics* $\gamma_1, \cdots, \gamma_k$ *may not be systoles.* This is the main result proved in [Th].

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